5.1 Introduction to the energy spectrum

While the equations are exact, the method is limited. Replace the time series of water levels of duration $\tau$ by a Fourier series:

$$\eta = \sum_{i} a_{i} \cos \omega_{i} t$$

The energy density per unit area of sea surface, $E$, is given by:

$$E = \rho g \frac{1}{2} \int \left| \eta \right|^{2} dt$$

$$= \rho g \frac{1}{2} \sum_{i} \int \left| a_{i} \right|^{2} \cos^{2} \omega_{i} t dt$$

$$= \rho g \frac{1}{2} \sum_{i} \int \left| a_{i} \right|^{2} \frac{1}{2} + \frac{1}{2} \cos 2 \omega_{i} t dt$$

$$= \rho g \frac{1}{2} \sum_{i} \left| a_{i} \right|^{2} + \rho g \frac{1}{2} \sum_{i} \frac{H_{i}^{2}}{2}$$

Thus $H_{i} = 2a_{i}$ is the height of a component wave of frequency $\omega_{i}$, and the spectrum is

$$S(\omega_{i}) = \frac{H_{i}^{2}}{8 \pi}$$
5 PROBABILITY & SPECTRA

5.2 Spectral Relationships

The Jonswap spectrum

The JONSWAP spectrum (Hasselmann et al. 1973) was developed for fetch-limited seas. It is the most widely used formula.

\[ Sopp(f) = a_0 g^2 (2\pi)^3 f^{-5} \exp \left[ -\frac{1}{2} (f/f_m)^3 \right] \gamma^2 \]

\[ f_m \] is the frequency at which the spectrum peaks:

\[ f_m = \frac{g f_m}{2.84} \left( \frac{g E}{U_0^2} \right)^{0.3} \]

\[ a_0 = 0.0330 \left( \frac{f_m U_0}{g} \right)^{0.5} \]

\[ a = \exp \left[ -\left( f - f_m \right)^2 / 2 \sigma^2 f_m^2 \right] \]

\[ \gamma = 0.09 \quad f > f_m; \]

\[ \gamma = 0.07 \quad f < f_m \]

\( \gamma \) is the frequency at which the spectrum peaks:

\( \sigma = 0.09 \quad f > f_m; \)

\( \sigma = 0.07 \quad f < f_m \)

For \( \gamma = 1 \) JONSWAP equation reduces to the Pierson-Moskowitz.

For \( \gamma > 1 \) JONSWAP spectrum has a higher and narrower peak than the PM, as is observed in nature.

\( \gamma = 3.3 \) was found to give a good fit to data from the North Sea.

Lewis and Allos (1990) have attempted to set consistent parameter values for the JONSWAP equation, but they are frequently misunderstood or misused.

Huang et al. (1981) used results of laboratory studies which showed that the spectrum of a single pure water wave had a second peak at a higher frequency. They used this to deduce the following 2-parameter spectrum.

\[ S(f) = \phi \left( \frac{f}{f_m} \right)^n \exp \left[ -\left( \frac{f}{f_m} \right)^2 / \left( 2 \sigma^2 f_m^2 \right) \right] \]

where \( \phi \) and \( n \) are functions of the wave field, which are to be related to the gross parameters of the wind and fetch.

Hinwood et al. (1983) have shown this spectrum to provide a better fit to Bass Strait swell data and bimodal (sea + swell) data than other spectra and have modified it for water of transitional depth.

5 PROBABILITY & SPECTRA

5.2 Spectral Relationships

The Wallops spectrum

Huang et al. (1981) used results of laboratory studies which showed that the spectrum of a single pure water wave had a second peak at a higher frequency. They used this to deduce the following 2-parameter spectrum.

\[ S(f) = \beta \left( \frac{f}{f_m} \right)^n \exp \left[ -\left( \frac{f}{f_m} \right)^2 / \left( 2 \sigma^2 f_m^2 \right) \right] \]

where \( \beta \) and \( n \) are functions of the wave field, which are to be related to the gross parameters of the wind and fetch.

Hinwood et al. (1983) have shown this spectrum to provide a better fit to Bass Strait swell data and bimodal (sea + swell) data than other spectra and have modified it for water of transitional depth.

5 PROBABILITY & SPECTRA

5.3 Directional Wave Spectra

Particularly within the fetch, wave energy is distributed in all directions. This spread may be described by the directional spectrum \( S(\theta, \omega) \), where \( S(\theta, \omega) \) is a measure of the energy propagating in the directional band \( \theta \) to \( \theta + d\theta \) and in the frequency band \( \omega \) to \( \omega + d\omega \). The total energy is given by

\[ S(\theta, \omega) = D(\theta) G(\omega) \]

The spreading function, \( D(\theta) \), is often taken to be of the form

\[ D(\theta) = \cos^n(\theta) \]

Early studies suggest that \( n = 1 \) or 2, while more recent studies have found values at the end of the fetch as high as 16 and far from the fetch as high as 80.

Mitsuyasu (1975) proposed

\[ n = 11.5 (c_m U / g)^{2.5} \]

where \( c_m \) is the celerity of the spectral peak component.
5 PROBABILITY & SPECTRA

5.4 Shallow Water Wave Spectra – Young and Verhagen

Based on data from an extensive series of field experiments in Lake George, near Canberra, Young and Verhagen derived relationships for the growth of wind waves in shallow water and finite fetches.

Dimensionless variables are based on the wind speed, $U_{10}$, the still water depth, $d$, the fetch length, $x$, the total wave energy, $E$, and the peak frequency, $f_p$.

- Non-dimensional depth: $\delta = g d / U_{10}^2$
- Non-dimensional fetch: $\chi = g x / U_{10}^2$
- Non-dimensional frequency: $\nu = f_p U_{10} / g$
- Non-dimensional energy: $\varepsilon = \sigma_{rms}^2 / U_{10}^2$


5.5 Distribution of Wave Heights

The heights of successive waves in a real sea may be found from a wave record of about 20 min duration.

The frequency distribution has a clear mode and a long tail showing that very high waves may occur but with very low frequency.

The probability density function $p(H)$ is the probability that a wave of height $H$ will occur. $P(H)$ is the cumulative probability that a given wave has height $> H$.

\[ p(H) = \frac{2 H}{H_{rms}} \exp \left( - \frac{H}{H_{rms}} \right) \]

Wave data fit the Rayleigh distribution very well. However, some analyses have shown a tendency for very low and very high waves to occur slightly less frequently than predicted.

The Rayleigh distribution is based on the envelope definition of wave heights while data are usually based on the zero upcrossing definition, largely accounting for these slight differences.
5 PROBABILITY & SPECTRA

5.5 Distribution of Wave Heights

The significant wave height, \( H_{1/3} \) or \( H_s \) was initially defined as the highest typical wave in the record, now defined as the mean of the highest one-third of all waves in a record:

\[
H_{1/3} = \frac{1}{3} \sum_{i=1}^{3} p(H) dH
\]

It is closely related to the wave height identified by visual estimation.

Based on the Rayleigh distribution we can relate different measures of wave height:

- \( \sigma_{\eta} \) = standard deviation of water level
- \( H_{rms} \) = root mean square wave height
- \( \eta \) = mean wave height
- \( H_{1/3} \) = significant wave height
- \( \sqrt{2} H_{rms} = 4 \sigma_{\eta} \)

Different measures of wave period are in use.

The corresponding frequencies are usually evaluated from the spectrum and use of the Rayleigh distribution, rather than doing a zero crossing analysis.

\[
\begin{align*}
\eta_{mn} & = m_0 / m_2 \\
\tau_{mn} & = m_0 / m_1 \\
\tau_{n} & = 2 \pi / \omega_n \\
\end{align*}
\]

where \( m_n \) is the \( n \)th moment of the spectrum.

\[
\begin{align*}
T_\eta / T_\tau & = 1.41 \\
T_\eta / T_\tau & = 1.29
\end{align*}
\]