From Spheres to Circular Cylinders: Classification of Flow Transitions and Structure of Bluff Ring Wakes

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Following on from a previous study of the linear stability of the wakes of bluff rings, the structure of the saturated non-axisymmetric transition modes is investigated numerically.

Asymmetric simulations verify the results from a previous linear Floquet stability analysis performed on the wakes of bluff rings. The asymmetric wake corresponding to every predicted asymmetric instability mode over the entire aspect ratio range is successfully computed, and isosurface plots are presented elucidating the vortical structure of the asymmetric wakes. The existence of three asymmetric transition mode regimes (Modes I, II and III) for aspect ratios $Ar \lesssim 3.9$ is verified, and the appearance of the vortex shedding modes A, B and C is also correctly represented through asymmetric computations. The topology and spatio-temporal symmetry of the vortex shedding wakes correspond to the Mode A and Mode B transitions of the straight circular cylinder wake, and the additional Mode C instability is shown to correspond to a subharmonic asymmetric wake with an azimuthal wavelength approximately two diameters ($2d$).

Criticality of the transition modes is determined by modeling the transitions with the Landau equation. The regular Mode I and Mode III transitions are found to occur through supercritical and subcritical bifurcations, respectively, and the secondary Hopf bifurcations to these transitions, as well as the Mode II Hopf transition, are found to be supercritical. We verify that the Mode A and Mode B transitions are subcritical and supercritical, respectively, and we determine that the criticality of the Mode C transition is supercritical. The Landau constant is evaluated in the vicinity of the Hopf transitions.

1. Introduction

The route to turbulence associated with increasing Reynolds number for the wakes of bluff bodies has been the subject of a significant body of research over many years. Simple geometries such as the sphere (Taneda 1956; Magarvey & Bishop 1961b; Johnson & Patel 1999; Tomboulides & Orszag 2000), the circular cylinder (Williamson 1988a, 1996), and rectangular cylinders (Hourigan, Thompson & Tan 2001; Mills, Sheridan & Hourigan 2002, 2003) have all been studied to gain an understanding of the intermediate flow states and the transitions that can be found between the steady, attached, laminar wakes of low Reynolds number flows (typically $Re < 5$ for circular cylinder wakes), to the chaotic, turbulent wakes observed at higher Reynolds numbers ($Re > 500$ for the circular cylinder wake). The remnants of such transitions often persist even in fully developed turbulent flow. For example, Strouhal shedding and the Mode B spanwise wavelength...
persists up to high Reynolds numbers for circular cylinder wakes (Wu, Sheridan, Welsh & Hourigan 1996).

The intermediate wake states observed are due either to linear instabilities of the flow field, such as the three-dimensional vortex shedding modes of the circular cylinder (Barkley & Henderson 1996), or from global effects such as oblique shedding modes (Williamson 1988a, 1989).

The boundary layer transition from attached to separated flow has been observed to occur prior to the linear instability modes of the wakes of spheres and discs. As this transition marks the beginning of the formation of the recirculating flow behind a bluff body, which in turn becomes unstable to the various modes studied here, it is of underlying importance. Tomboulides & Orszag (2000) found the transition to separated flow to occur in the wake of the sphere at \( Re = 21 \), and the cylinder transition at \( Re = 6 \) was predicted by Noack & Eckelmann (1994b).

The linear instabilities that arise in the wake behind the sphere have been studied by Natarajan & Acrivos (1993). They predicted that the wake behind the axisymmetric sphere and disc undergoes a regular (steady-steady) asymmetric transition, followed by a Hopf (unsteady) transition at higher Reynolds numbers. Both these transition modes have \( m = 1 \) azimuthal symmetry (only one wavelength of the unstable mode around the circumference of the body). Both experimental observations (Magarvey & Bishop 1961b, a; Magarvey & MacLatchy 1965), and asymmetric numerical simulations (Johnson & Patel 1999; Tomboulides & Orszag 2000; Thompson, Leweke & Provansal 2001a) support these findings for the sphere wake, as well as the predicted critical Reynolds number for the initial regular transition. The asymmetric wake structure of the sphere wake following the initial asymmetric transition is characterised by a radial shift of the recirculation bubble behind the sphere from the axis, and the development of two trails of opposing streamwise vorticity stretching far downstream positioned either side of a plane of symmetry along the axis (Johnson & Patel 1999). Just prior to the Hopf bifurcation there is a deformation of the twin trails of vorticity, being drawn closer together (Thompson, Leweke & Provansal 2001a) near the end of the recirculation bubble. Following the secondary transition the bubble becomes unstable and sheds vorticity into the wake, forming hairpin shaped vortices shedding on opposite sides of the axis every half period (Tomboulides & Orszag 2000). A plane of symmetry in the wake is maintained following the secondary Hopf transition (Mittal 1999) until \( Re > 375 \).

In order to study geometric effects on bluff body wake transition, the wakes from tori or bluff rings oriented normal to the flow are studied. Figure 1 shows a schematic diagram of the bluff ring system. By varying the single geometric parameter \( Ax \), a uniform axisymmetric body is described varying from a sphere at \( Ax = 0 \), to a straight cylinder in the limit \( Ax \to \infty \).

Monson (1983) studied the drag and vortex shedding behind rings of various diameters falling through a fluid. A variety of shedding patterns were observed and, as the ring minor cross-section became smaller with respect to the outer diameter of the ring, the shedding patterns and drag characteristics approached those observed behind the straight circular cylinder. This characteristic of the bluff ring geometry partially motivated an experimental study by Leweke & Provansal (1995), who used rings of large aspect ratio to approximate the wake behind a circular cylinder, but without the end effects that are difficult to avoid with straight cylinder sections. Despite their mention of the discontinuities present in the bluff ring Strouhal number profiles due to asymmetric shedding modes arising in the flow field, no visualisation of such modes was attempted. For smaller aspect ratios (\( Ax < 6 \)), Bearman & Takamoto (1988) observed the wake behaviour of the bluff rings to more closely resemble the sphere and disc than the cylinder.
The point in aspect ratio parameter space at which the ring wake switches from a circular cylinder-type vortex shedding to a sphere-type hairpin wake was suggested by Monson (1983) to occur at an aspect ratio of approximately $Ar = 4.5$. The recent well-resolved numerical study of Sheard, Thompson & Hourigan (2001) predicts this switch to occur at an aspect ratio of approximately $Ar \approx 3.9$.

An attempt was made to study the vortex shedding modes for bluff rings by application of a linear stability analysis scheme by Sheard et al. (2001, 2003). The length scale for Reynolds number calculations is the diameter of the ring cross-section, $d$, consistent with the sphere and cylinder length scales used in previous studies. The analysis found evidence of modes of vortex shedding analogous to both the Mode A and Mode B transition modes identified in the wake of the circular cylinder experimentally (Williamson 1988a), and by numerical stability analysis (Barkley & Henderson 1996; Thompson, Leweke & Williamson 2001b). The critical Reynolds numbers for the Mode A and Mode B transitions in the wakes of bluff rings varied over the ranges $188 < Re < 200$ and $258 < Re < 300$ respectively. The azimuthal wavelengths for the modes were found to be approximately $3.9d$ for the Mode A instability, and $0.8d$ for the Mode B instability. Sheard et al. (2001, 2003) found evidence of a third asymmetric shedding instability with an azimuthal wavelength of $1.6d < \lambda < 1.7d$. This wavelength, and the observed 2T symmetry of the mode, led to the conclusion that it was analogous to the Mode C transition observed in the perturbed circular cylinder wake (Zhang et al. 1995), and the Mode S wake behind the square cylinder (Robichaux, Balachandar & Vanka 1999).

Rings with smaller aspect ratios have been shown to exhibit wakes vastly different to the straight circular cylinder (Roshko 1953). He observed a similar decrease in frequency between bluff ring wakes and the circular cylinder wake to that quantified by Leweke & Provansal (1995). Furthermore, he observed a significant change in the Strouhal-Reynolds number profiles for much smaller rings. The aspect ratios at which this change occurred was smaller than those studied by Leweke & Provansal (1995), however dye visualisations of Monson (1983) support the finding of Roshko (1953).

The tori with smaller aspect ratios studied by Monson (1983) gave an indication that a transition to an unsteady mode may occur resulting in a wake similar to the hairpin shedding observed behind the sphere (Tomboulides & Orszag 2000; Johnson & Patel
Stability analysis of bluff rings with small aspect ratio \((Ar < 4)\) by Sheard et al. (2001, 2003) has shown that the transition modes of the sphere are in fact observed for a variety of bluff rings in the aspect ratio range \(0 \leq Ar < 1.6\). Three distinct transition modes (Mode I, Mode II and Mode III) have been identified over the aspect ratio ranges \(0 \leq Ar < 1.6, 1.6 \leq Ar < 1.7\) and \(1.7 < Ar \leq 3.9\), respectively. Furthermore, Modes I & III are predicted to be regular transitions, and are followed at higher Reynolds number by Hopf bifurcations, and Mode II is predicted to be a spontaneous asymmetric Hopf bifurcation to the axisymmetric base flow. The Mode I instability is predicted to occur at Reynolds numbers decreasing from \(Re = 211\) at \(Ar = 0\) to \(Re = 72.6\) at \(Ar = 1.4\) (Sheard et al. 2003). A Hopf transition occurs over the same aspect ratio range at higher Reynolds numbers. The dye visualisations of the ring wakes from Monson (1983) at high solidity (small aspect ratio) show this trend, with periodic asymmetric wakes being observed for the ring with \(Ar = 2.596\) at \(Re = 115\). Interpolating the critical Reynolds number profiles from Sheard et al. (2003) show that the Hopf bifurcation would occur in the Reynolds number region \(100 < Re < 105\), in agreement with the existing experimental observations.

The non-linear behaviour of the asymmetric transitions in the wakes of bluff rings is modelled here using the Landau model (Provansal, Mathis & Boyer 1987). The Landau model and its application are described in §2.

2. Transition Behaviour

The Landau model provides a means for studying the non-linear behaviour near the transition Reynolds number. It has been used widely in describing and classifying bluff body wake transitions previously: for example, the circular cylinder wake Hopf bifurcation (Dušek, Fraunié & Le Gal 1994; Zielinska & Wesfreid 1995); the circular cylinder Mode A and B transitions (Henderson 1997); and the sphere wake transitions (Ghidersa & Dušek 2000; Thompson, Leweke & Provansal 2001a; Provansal & Ormières 1998).

Landau & Lifshitz (1976) proposed the model to describe the growth and saturation of the perturbation post-transition. The governing equation is

\[
\frac{dA}{dt} = (\sigma + i\omega)A - l(1 + ic)|A|^2A + \ldots
\]

Here, \(A(t)\) is a complex variable representing the mode amplitude. A description of this equation, and its place in the history of stability analysis, is provided in Provansal et al. (1987). Here, we review briefly the characteristics of the model and the method of its application pertinent to the present study.

The complex variable \(A(t)\) represents the amplitude of the perturbation mode from the base flow. The right-hand side of the equation gives the first two non-zero terms of a series expansion. Provided \(l\) is positive, these first two terms should provide a good description of the non-linear behaviour in the neighbourhood of the transition, since the saturated amplitude should still be small. This is not true if \(l\) is negative; in that case, the cubic term accelerates the growth of the perturbation and quintic or even higher-order terms are required for saturation. Thus, the sign of \(l\) plays an important role in classifying the transition. Positive \(l\) means the transition is supercritical (non-hysteretic), while negative \(l\) means it is subcritical (hysteretic). The parameter \(\sigma\) is the linear growth rate of the perturbation, thus at the transition point, its value changes from negative to positive. Also, \(\omega\) is the angular oscillation frequency during the linear growth phase, this is non-zero for a Hopf bifurcation. The parameter \(c\) is called the Landau constant. It is a non-dimensional quantity (unlike \(l\)) and hence does not depend on the position in the
wake where $A(t)$ is measured. It modifies the oscillation frequency at saturation, and in addition, its size and magnitude determine global behaviour in related flow systems such as wake behaviour under external oscillatory forcing (Le Gal, Nadim & Thompson 2001).

For regular transitions in a fluid flow (i.e. steady to steady transitions) the amplitude $A(t)$ is a real variable and the imaginary coefficients ($\omega, c$) are zero, simplifying the analysis. A flow transition to a time-dependent state, such as a Hopf bifurcation, requires phase information. In this case the imaginary coefficients are non-zero and the amplitude is required to be a complex variable.

The usual way to manipulate the Landau equation (i.e. Dušek et al. 1994) is to write the complex amplitude variable as

$$A = \rho \exp(i\Phi).$$

(2.2)

Here, $\rho \equiv |A|$ is a real variable describing the mode amplitude, and $\Phi$ is a real variable providing the phase of the mode. The Landau equation can then be split into real and imaginary parts:

$$\frac{d \log(\rho)}{dt} = (\sigma - \gamma \rho^2 + \ldots),$$

(2.3)

$$\frac{d \Phi}{dt} = (\omega - l p \rho^2 + \ldots).$$

(2.4)

Using equation (2.3) and noting that at saturation the (real) amplitude will not change in time, gives $\rho_{sat}^2 = \sigma / l$. In addition, since $\sigma$ is necessarily proportional to the Reynolds number increment above critical in the neighbourhood of a simple transition, the energy in the mode (proportional to $\rho^2$) varies as $Re - Re_c$, where $Re_c$ is the critical Reynolds number. This behaviour has been verified numerically (e.g. Dušek et al. 1994) and experimentally (e.g. Ormieres & Provansal 1999) for different supercritical wake transitions. The equation for the phase (2.4) also provides useful information. If the flow reaches a periodic state at saturation, $d\Phi/dt$ becomes the constant angular frequency of oscillation ($\omega_{sat}$). It takes the value $\omega_{sat} = \omega - l p_{sat}^2 = (\omega + \sigma c)$. Hence, $\sigma c$ determines the shift from the oscillation frequency in the linear regime, as the flow saturates.

Given equation (2.3), it is possible to determine the real parameters in the model from numerical computations by plotting $d \log |A|/dt$ against $|A|^2$. The y-intercept gives the linear growth rate $\sigma$ and the slope of the curve for small amplitudes corresponds to the $-l$. For the truncated cubic Landau model to describe the transition well, equation (2.3) indicates the plot should be linear. Plotted this way, the time trajectory of the transition should be start at the $y$-axis, and finish on the $x$-axis if the flow reaches a periodic or steady asymptotic state. If the slope of the curve is positive at the $y$-axis, the transition is subcritical and at least quintic terms are required in the Landau equation to describe the saturation process with any accuracy. For those cases where the cubic model is adequate, it is possible to determine the Landau constant by measuring the oscillation frequency of the perturbation in the linear regime and at saturation, and by rearranging equation (2.4) to form equation (2.5):

$$c = \frac{\omega_{sat} - \omega}{\sigma}.$$  

(2.5)

From the numerical point of view, the derivative $(d \log |A|/dt)$ can be accurately estimated using finite-differences. For transitions to a time-dependent final state, the signal at small times is initially sinusoidal multiplied by an exponential growth factor. At larger times, as the flow saturates, the amplitude envelope asymptotes to a constant width and the sinusoidal oscillation frequency adjusts slightly according to equation (2.4). Thus, in the more complicated case of transition to a time-dependent final state, the derivative
can be estimated by finite-differences by using the heights of the approximately sinusoidal peaks and troughs during the growth and saturation of the instability.

To proceed further, the amplitude variable $A$ needs to be specified. Previous studies have taken different approaches. For example, Dušek et al. (1994), and Thompson et al. (2001a) used the transverse velocity component at a fixed point on the centreline of the circular cylinder wake. Zielinska & Wesfreid (1995) instead used the maximum transverse velocity component on the centerline. The position at which this occurs varies with Reynolds number. Henderson (1997) used the $L_2$ norm of spanwise velocity component for examining two- to three-dimensional transitions for the circular cylinder wake. However, because the numerical domain was necessarily truncated downstream before the mode amplitude decayed to zero, this also was not a unique global measure. For the analysis presented in this paper, the first method, as described in Thompson et al. (2001a), is adopted. This relies on the assumption that at any point in the near wake the transition behaviour will be representative of the global behaviour of the flow field. This enables the behaviour of a transient at a single point to be monitored, rather than calculating a global quantity in addition to evolving the flow. All the transitions studied in this paper are asymmetric transitions. The azimuthal velocity component at a point in the wake is nonzero prior to transition and thus captures the growth and development of the asymmetric mode. Thus the azimuthal component of velocity at a point in the wake is monitored to study the non-linear saturation of the asymmetric mode.

Asymmetric simulations were performed using axisymmetric flow fields as an initial condition. The axisymmetric flow fields were perturbed by the addition of a small random asymmetric velocity fluctuation with an amplitude of the order $10^{-4}$ of the mean flow. The flow fields were then evolved to saturation, and the time history of the azimuthal velocity component at a point in the near wake was recorded for post processing.

3. Numerical Method

3.1. The Spectral-Element Method

A spectral-element method was used for the numerical simulations in this investigation. This method is described in detail in Karniadakis (1990), and Thompson et al. (1996), and is based on the Galerkin finite-element method. Tensor products of high-order Lagrange polynomials are used to construct the shape functions within each element. The node points of the Lagrangian polynomials correspond to the Gauss-Legendre-Lobatto quadrature points, allowing accurate and efficient integration over each element. Asymmetric simulations of the flow fields are performed using a Fourier expansion in the azimuthal direction.

3.2. Validation of the Spectral-Element Code and Grid Resolution Studies

The numerical spectral-element software used here has been validated previously and employed successfully by Thompson et al. (1996), and Sheard et al. (2001, 2002, 2003) in related wake studies. The resolution of the meshes constructed to model the bluff ring geometries are consistent with similar numerical studies of both the sphere (Tomboulides & Orszag 2000; Thompson et al. 2001a) and the circular cylinder (Barkley & Henderson 1996). Upstream of the circular ring cross-section, a uniform velocity inlet boundary is imposed. This uniform velocity is also imposed on the side boundaries, and zero normal velocity gradient is imposed at the outlet.

Grid resolution studies (Sheard et al. 2001, 2002, 2003) determined domain sizes, the number of mesh elements, and the required number of nodes per element to resolve flow field characteristics to within 1%. The meshes typically consisted of 380 to 400 elements,
with 64 (8 × 8) nodes per element. To properly resolve the vorticity field for \( Re > 300 \), 9 × 9 node elements were employed. The inlet length, transverse length, and outlet length were 15, 30 and 25 ring cross-section diameters, respectively.

4. Results

4.1. Asymmetric Transition Modes at Small Aspect Ratios

This section presents flow visualisation plots to illustrate the asymmetric vortical structures present in the wakes of bluff rings at small aspect ratio. Values of aspect ratios were selected to isolate each of the three asymmetric transition modes identified in previous work. As the Mode I transition has been predicted to occur for the wide aspect ratio range \( 0 \leq Ar < 1.6 \), two aspect ratios in this range are studied here to capture the wake structures that occur following the Mode I transition. In particular, this range spans the change in topology as the sphere transforms to a ring with a hole. The aspect ratios considered for the Mode I transition are \( Ar = 0.6 \) and 1.2. For the Mode II transition \( Ar = 1.6 \) is used, and \( Ar = 2.0 \) is used to study the Mode III transition.

4.1.1. Asymmetric Wake Structure of the Transitions

The saturated asymmetric wakes following the Mode I transition are represented by streamwise vorticity isosurfaces plots in figure 2. For comparison, the same visualisation for the wake of the sphere (\( Ar = 0 \)) is also included. Flow is from the upper right corner to the lower left corner in each case. The ring/sphere is visible at the upper right of each frame. The light and dark isosurfaces represent negative and positive streamwise vorticity respectively, highlighting significant streamwise vorticity in the wake. It is apparent from these isosurface plots that a plane of symmetry exists in the wake through the centre of the ring. This symmetry has also been observed for the sphere (Tomboulides & Orszag 2000; Johnson & Patel 1999). Furthermore, this Mode I transition has \( m = 1 \) azimuthal symmetry, i.e. there is a single azimuthal wavelength spanning the azimuthal domain \( (2\pi) \). The similarity of figures 2(b) and 2(c) indicate that the flow fields are indeed both products of the growth and saturation of the previously predicted Mode I transition, and that the emergence of a small hole on the axis of the ring in this transition regime does not significantly modify the transition process. Note the wings of streamwise vorticity immediately behind the ring, and the pair of vorticity tails stretching far downstream. The latter correspond to the classic “double-threaded wake” observed in the wake of the sphere (Tomboulides & Orszag 2000; Thompson et al. 2001a; Magarvey & Bishop 1961b, a) following the first asymmetric transition.

The saturated wake structure following the Mode II transition is shown in figure 3, again visualised using streamwise vorticity. The near wake in this case is not dissimilar to the near wake region for the Mode I transition directly behind the ring, with wings of streamwise vorticity of opposing sign wrapped around longer tails of streamwise vorticity extending downstream. However, rather than forming the stationary double-threaded wake of the Mode I transition, the Mode II transition wake sheds long slanted pairs of vortices of alternating sign aligned with the axis downstream. The Mode II transition is a Hopf transition from a steady axisymmetric flow to an asymmetric unsteady flow. This is in close agreement with the predictions of the stability analysis of Sheard et al. (2001, 2003) for this transition, which indicated that the most amplified Floquet mode was periodic, corresponding to a Hopf bifurcation to a periodic asymmetric wake. Note that the planar symmetry and the \( m = 1 \) azimuthal symmetry, observed for the Mode I transition, are retained for the Mode II transition.

The wake of the ring following the Mode III transition is shown in figure 4. This mode
is a regular (steady-steady) transition to asymmetry, as is Mode I; however structurally it is vastly different to the Mode I transition wake. No asymmetry is observed along the streamwise axis for the Mode III transition, rather the bands of vorticity present in the wake are located directly downstream of the ring body. The structures indicate that the transition involves a loss of stability of the axisymmetric recirculation rings behind the bluff ring body. This agrees with the previous stability analysis of Sheard et al. (2003).

The Mode III transition is observed for rings with aspect ratios $Ar < 4$. At larger aspect ratios ($Ar \geq 4$) the asymmetric transition is preceded by a Hopf bifurcation to a periodic axisymmetric wake, similar to the von Kármán vortex street in the wake of a circular cylinder. The $m = 1$ azimuthal symmetry of the Mode III transition is not maintained throughout its aspect ratio range $1.8 \leq Ar < 4$. Stability analysis of the wake for the $Ar = 3$ ring predicts an azimuthal symmetry of $m = 2$, that can be observed in figure 4(b). The $m = 2$ azimuthal symmetry creates two perpendicular planes of symmetry intersecting along the axis. The azimuthal symmetry of the wake in figure 4(a), corresponding to a smaller aspect ratio, is still $m = 1$.

To further illustrate the changes in azimuthal symmetry and flow structure for the Mode III transition with increasing aspect ratio, the isosurfaces plots from figure 4 are shown in figure 5 with the wake viewed from directly behind the ring. Note in figure 5 the structural similarity between the Mode III wake for the two rings. If one were to imag-
Figure 4. Streamwise vorticity isosurface plots of the $Ar = 2$ ring at $Re = 100$ (a) and the $Ar = 3$ ring at $Re = 115$ (b) following the Mode III transition. Note the change in symmetry between the two aspect ratios. Isosurface contours as per figure 2.

ine making a cut through the ring in figure 5(a) at the upper section, and azimuthally stretching the ring and the wake into a semicircle, the shape of the asymmetric structures would almost exactly match those present over half the ring in figure 5(b). This observation supports the prediction that these flow fields both evolve from the same transition mode. Furthermore, it suggests that this mode scales with the length scale for the cross-section of the ring (d), rather than the ring as a whole (D). If the steady asymmetric transition were not impeded by the axisymmetric Hopf bifurcation of ring wakes with $Ar > 4$, then it is expected that a Mode III transition would be found for all aspect ratios, with an azimuthal wavelength based on the ring cross-section diameter, d, approaching the spanwise wavelength for a regular asymmetric transition for a straight circular cylinder with unsteady modes suppressed. Noack & Eckelmann (1994a) found evidence of a regular three-dimensional transition mode to a steady circular cylinder wake using a global low-dimensional Galerkin method. The predicted mode had a spanwise wavelength of $\pi d$, and was neutrally stable at $Re = 200$. The sheer impracticality of the mode for a circular cylinder wake meant that it was not studied further, however it is interesting to compare the wavelength of that mode with the azimuthal span of the Mode III transition presented here. Based on the ring cross-section diameter, d, the Mode III transition occurs over spans of 6.28d and 4.71d for $Ar = 2$ and $Ar = 3$ respectively. It is feasible that these wavelengths are tending towards the limiting spanwise wavelength of $\pi d$ for the regular three-dimensional transition mode of the circular cylinder.

4.2. Landau Modelling of the Regular Mode I & III Asymmetric Transitions

The non-linear behaviour of the asymmetric transition modes for rings with aspect ratios $Ar < 4$ is investigated in the following sections. As described previously, the Landau equation is assumed to model the growth of asymmetric transients in the wake of the rings at Reynolds numbers near to the transition point. The criticality of each mode is determined, and where applicable comparisons are drawn to previous work on sphere stability (Thompson et al. 2001a), and (linear) ring stability Sheard et al. (2003), and circular cylinder wake stability (Henderson 1997). Analysis of the growth of the azimuthal velocity component at a point in the wake directly downstream of the ring allows the coefficients of the Landau equation to be determined. The regular transitions, Mode I and Mode III will be treated below. The Hopf transition Mode II, and the Hopf transitions following the Mode I and Mode III regular transitions will be analysed after the asymmetric flow structures of those modes are described.
Figure 5. Isosurface plots of streamwise vorticity indicating \( m = 1 \) and \( m = 2 \) azimuthal symmetry of the Mode III transition for the \( Ar = 2 \) (a) ring and \( Ar = 3 \) (b) ring respectively. Flow fields viewed from directly behind rings, and isosurface shading is as per figure 2.

Figure 6. (a) Left: Growth and saturation of the asymmetric transient in the wake of the \( Ar = 0.6 \) ring at \( Re = 130 \). (b) Right: Amplitude derivative versus amplitude squared plot. Y-axis intercept gives growth rate, \( \sigma \), and gradient close to y-axis gives saturation term, \( I \). The negative slope and linear behaviour near the y-axis indicate that the transition is supercritical.

4.2.1. The Mode I Transition

The Mode I transition is found to be supercritical for \( Ar = 0.6 \) and \( Ar = 1.2 \). This is in agreement with the work by Ghidersa & Dušek (2000) and Thompson et al. (2001a) on the corresponding sphere wake transition (the sphere is defined by the ring with \( Ar = 0 \), and is a limiting case for the Mode I transition regime). The criticality was determined from the sign of \( l \) in the Landau model and the behaviour of the transient perturbation at small amplitudes as described in §2.

The critical Reynolds numbers for the transitions are estimated from the growth rate, \( \sigma \), variation with Reynolds number. The resulting critical Reynolds numbers matched those predicted by the linear Floquet stability analysis of Sheard et al. (2003) to within 0.5%, with values of \( Re_c = 114 \) for \( Ar = 0.6 \), and \( Re_c = 77.6 \) for \( Ar = 1.2 \) being
determined. For both aspect ratios similar transient behaviour was obtained, indicating that the Mode I transition is supercritical for all aspect ratios. Examples of the plots used to determine the coefficients of the Landau model are provided in figure 6. These plots show the behaviour of the wake of the $Ar = 0.6$ ring at $Re = 130$, approximately 10% above the critical Reynolds number ($Re_c = 114.1$) determined from Floquet stability analysis (Sheard et al. 2003). Figure 7 gives the variation in growth rate with Reynolds number for the $Ar = 0.6$ ring from figure 6. Both the previous Floquet analysis and an independent quadratic fit to the current data provide a critical Reynolds number at this aspect ratio of $Re_c = 114.1$.

4.2.2. Mode III Transition

The Mode III transition was predicted to be a regular transition (Sheard et al. 2003). Determination of $l$, shows that the transition is subcritical. The behaviour of this transition for a ring with $Ar = 2$ at $Re = 98$ is shown in figure 8. Figure 8(b) shows the subcritical behaviour, with a distinct non-linearity and positive gradient at the $y$-intercept, indicating the necessity for higher order terms to be included in the Landau equation to adequately model the non-linear behaviour of the transition.

4.3. Visualising the Hopf Bifurcation Modes of Small Aspect Ratio Ring Wakes

This section provides visualisations of the wakes of small aspect ratio rings at Reynolds numbers above the critical Reynolds numbers for the Hopf transition. The method of Jeong & Hussain (1995) is used capture the asymmetric vortical structures observed in the wakes.

Over the aspect ratio range $0 < Ar < 4$, the regular asymmetric Mode I and Mode III transitions are followed by a secondary Hopf transition to unsteady flow at higher
Reynolds numbers. For the sphere wake this transition occurs at $Re = 272$ (Thompson et al. 2001a; Johnson & Patel 1999), and the resulting wake is characterised by a planar symmetry with $m = 1$ azimuthal symmetry. The wake structure consists of hairpin shaped vortex loops shedding alternately from opposite sides of the axis. It is shown that hairpin wakes are observed subsequent to both the Mode I and Mode III transitions. The Mode II transition is a Hopf transition, and the wake produced following the transition is structurally analogous to the hairpin wakes observed for the Hopf transitions in both the Mode I and Mode III aspect ratio regimes. Landau modelling will be used to determine the criticality of the transitions in § 4.4.

The wakes observed following the transition to unsteady flow are visualised in figure 9. Note in figure 9 that the structure of each wake is similar with the exception of figure 9(d). Figures 9(a–e) all show an azimuthal symmetry $m = 1$, and a plane of symmetry along the axis. Figure 9(f) shows an azimuthal symmetry $m = 2$, and two perpendicular planes of symmetry. This symmetry is consistent with the symmetry of the regular Mode III transition at this aspect ratio ($Ar = 3$). Hairpin style vortex shedding similar to that previously observed for the sphere (Magarvey & Bishop 1961b,a; Magarvey & MacLatchy 1965; Johnson & Patel 1999; Tomboulides & Orszag 2000) is evident in each case. The sphere wake is provided in figure 9(a) for comparison. A difference in wake structure is observed for the rings in the Mode III regime (figures 9(e) and 9(f)), with vortical structures shedding from the inner surfaces of the ring near to the axis. These internal vortical structures become more complex as the bluff ring aspect ratio is increased (figure 9(f)).

The Mode II wake (figure 9(d)) differs from the other wakes in that the Strouhal frequency of the oscillation was not periodic, and the average Strouhal frequency obtained was far lower than for the other cases. The Strouhal frequency was found to be approximately 30% of the corresponding frequencies for the secondary Hopf transition wakes following the Mode I and Mode III transitions. The isosurface plot of the Mode II wake has been obtained from the same velocity field as the streamwise vorticity plot from figure 3. Structurally the Mode II wake is different to the other wakes. Instead of the hairpin wake, with vortices being shed either side of a plane of symmetry temporally phase shifted by half a period, long sheets of vorticity are cast into the wake from one side of the axis, in keeping with the observations of the underlying streamwise vorticity contours in figure 3. The predictions of linear stability analysis (Natarajan & Acrivos
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Figure 9. Wakes of sphere and bluff rings with aspect ratios $Ar = 0$ (a), $Ar = 0.6$ (b), $Ar = 1.2$ (c), $Ar = 1.6$ (d), $Ar = 2.0$ (e), $Ar = 3.0$ (f). Parts (a–c) represent unsteady wake transitions that follow the regular Mode I transition at $Re = 300$, $Re = 160$ and $Re = 120$ respectively. Part (d) represents the Mode II transition wake at $Re = 100$. Parts (e) and (f) represent the transition following the Mode III transition of the wake, at $Re = 150$ and $Re = 138$ respectively. Isosurfaces represent Hussain field to elucidate vortical structures in the wake. Note the plane of symmetry through centre of each ring, and similar vortical structure of the wakes. Flow is from top right to bottom left of each frame.

1993; Sheard et al. 2003) regarding the azimuthal mode numbers of these Hopf transition wakes are validated by the observations made from figure 9, with the wakes in parts (a–e) all exhibiting an azimuthal symmetry of $m = 1$, and part (f) exhibiting an $m = 2$ symmetry.

4.4. Criticality of the Hopf Transitions for Small Aspect Ratio Ring Wakes

The non-linear behaviour of the Hopf transitions of bluff rings with aspect ratios in the range $0 \leq Ar \leq 4$ is examined by application of the Landau equation as per the method described in § 2 and applied in § 4.2. The secondary Hopf transition following the regular Mode I transition will be analysed in § 4.4.1, the Mode II Hopf transition will be analysed in § 4.4.2, and the secondary Hopf transition in the Mode III transition aspect ratio regime will be analysed in § 4.4.3.
4.4.1. Secondary Hopf Transition of the Mode I Wake

The secondary transition for the sphere wake has been shown to be supercritical (Thompson et al. 2001a). The secondary transition to the Mode I transition behind bluff rings is also shown to be a supercritical Hopf transition. This is expected as the sphere lies in the Mode I transition aspect ratio regime. Simulations were performed to obtain Landau coefficients from the envelope of oscillation of speed at a point in the wake behind rings with $Ar = 0.6$ and $Ar = 1.2$.

To understand the behaviour of the velocity transient through the Hopf transition following the Mode I transition, plots showing both the evolution to saturation and the criticality behaviour are given in figure 10 for a ring with $Ar = 0.6$ at $Re = 140$.

The Landau constant, $c$, has been calculated in the vicinity of the secondary transition for the ring with $Ar = 0.6$. The value of the Landau constant was approximately $c = -0.5 \pm 0.05$. The Landau constant for the Hopf transition of a sphere wake ($Ar = 0$) has been shown to be $c = -0.55$ (Thompson et al. 2001a), close to the value we find for the $Ar = 0.6$ ring.

4.4.2. The Mode II Hopf Bifurcation

To model the non-linear behaviour of the Mode II transition, the Landau model was applied to the growth of the envelope of azimuthal velocity at a point in the wake approximately $4d$ downstream of the bluff ring cross-section. Landau coefficients were determined at Reynolds numbers $Re = 85$, 96, 98 and 100. The transition is found to occur through a supercritical bifurcation of the steady axisymmetric wake, resulting in an unsteady asymmetric wake. Plots showing the transient behaviour in the wake at $Re = 98$ are provided in figure 11. The plots in figure 11(b) shows that the transition is supercritical, with a negative gradient near the $y$-axis. The growth rate was determined from figure 11(b) to be $\sigma = 0.00709$. Note the small number of points resulting from the large period of oscillation of the Mode II instability. Despite this, seven data points exist over a linear growth regime ($250 \lesssim t \lesssim 1300$), sufficient for the present Landau modelling.

The Landau constant measured within the vicinity of the Mode II transition was found to be $c = -0.60$. This value was obtained at $Re - Re_c = 3.5$, for a ring with $Ar = 1.6$. 

Figure 10. (a) Left: Growth and saturation of the envelope of the asymmetric transient in the wake of the $Ar = 0.6$ ring at $Re = 140$. (b) Right: Amplitude derivative versus amplitude squared plot, showing supercritical behaviour.
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4.4.3. Secondary Hopf Bifurcation of the Mode III Wake

The previous sections have shown that the Hopf transition subsequent to the Mode I transition, and the Mode II Hopf transition, are supercritical and subcritical, respectively. The criticality of the Hopf transition that occurs in the ring wake following the Mode III transition is determined here to be supercritical. Plots of the envelope of the azimuthal velocity component at a point in the wake illustrate this behaviour. An example of the transient behaviour in the wake is given in figure 12.

The Landau constant was calculated for the secondary Hopf transition to the Mode III transition at aspect ratios of $Ar = 2$ and $Ar = 3$. For the ring with $Ar = 2$ the Landau constant is $c = -0.92$, and for $Ar = 3$ the Landau constant takes the value $c = -4.1$.

4.4.4. Variation of the Landau Constant with Aspect Ratio

In addition to finding the Landau constants for the secondary Hopf transitions of the wakes of rings with aspect ratios $Ar < 4$, a study of the axisymmetric Hopf transition of the wakes of rings with aspect ratios $Ar \geq 4$ has been performed. Landau constants are calculated for the Hopf transitions of wakes of rings with aspect ratios $Ar = 4, 5, 10, 20$ and 40.

A complete profile of the Landau constant variation with aspect ratio is presented in
Figure 12. (a) Left: Growth and saturation of the envelope of the asymmetric transient in the wake of the $Ar = 2$ ring at $Re = 99$. (b) Right: Amplitude derivative versus amplitude squared plot, showing supercritical behaviour, with Landau coefficients $\sigma = 0.014$ and $I = 1.5$ being determined.

Figure 13. The variation of Landau constants ($c$) with aspect ratio ($Ar$). Landau constants are calculated within the vicinity of the primary Hopf transition of the wake at each aspect ratio. The Landau constants computed over the asymmetric Hopf transition range are indicated by circles, and the axisymmetric Hopf transition range is represented by squares.

It is important to note the similarity between the Landau constants measured for both the $Ar = 0.6$ and $Ar \geq 5$, and the accepted values of the Landau constant for the sphere and straight circular cylinder ($c = -0.554$ and $2.6 < c < 3.0$, respectively).

The most interesting feature of this measured profile is significantly larger magnitude of the measured Landau constant at $Ar = 4$, near the crossover point in aspect ratio parameter space for the Hopf transition and asymmetric transition. The larger amplitude...
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<table>
<thead>
<tr>
<th>Aspect ratio (Ar)</th>
<th>Mode A</th>
<th>Mode B</th>
<th>Mode C</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(Re \geq 194.0, m = 4)</td>
<td>(Re \geq 301.4, m = 20)</td>
<td>(Re \geq 160.9, m = 10)</td>
</tr>
<tr>
<td>10</td>
<td>(Re \geq 194.3, m = 8)</td>
<td>(Re \geq 270.0, m = 40)</td>
<td>(Re \geq 222.1, m = 19)</td>
</tr>
<tr>
<td>20</td>
<td>(Re \geq 189.2, m = 16)</td>
<td>(Re \geq 261.2, m = 79)</td>
<td>(Re \geq 310.9, m = 38)</td>
</tr>
<tr>
<td>40</td>
<td>(Re \geq 187.9, m = 32)</td>
<td>(Re \geq 259.0, m = 153)</td>
<td>(Re \geq 338.7, m = 75)</td>
</tr>
</tbody>
</table>

Table 1. Results of the linear stability analysis on the vortex shedding wakes of bluff rings (see Sheard et al. 2003). The predicted critical Reynolds numbers and azimuthal mode numbers for the fastest-growing asymmetric vortex shedding modes are provided.

was due to a combination of a larger frequency shift through saturation of the wake, as well as a smaller growth rate of the instability than was measured for most of the other rings.

The \(Ar = 4\) ring marks a point in the parameter space where a significant alteration of the Landau constant profile occurs. For smaller aspect ratios (\(0 \leq Ar \leq 4\)), the Landau constant decreases continuously from \(c = -0.554\) to \(c = -9.8\). For larger aspect ratios (\(Ar > 4\)), the Landau constant remains relatively constant, decreasing from \(c = -2.37\) at \(Ar = 5\) to \(c = -2.55\) at \(Ar = 40\). These distinct Landau constant profiles correspond to the regions in parameter space where the Hopf transition is asymmetric (\(0 \leq Ar \leq 4\)), and axisymmetric (\(4 \leq Ar < \infty\)). This fundamental distinction in wake transition dynamics is the likely cause of the distinct Landau constant profiles.

4.5. Asymmetric Vortex Shedding Modes for Rings with \(Ar \geq 5\)

The existence of asymmetric shedding modes in the wakes of bluff rings has been predicted by stability analysis (Sheard et al. 2003). Experimental Strouhal profiles have also indicated asymmetric transitions in the vicinity of the critical Reynolds number for the Mode A and Mode B transitions for the circular cylinder (Leweke & Provansal 1995). The results of asymmetric simulations are provided here to validate the predictions of the earlier stability analysis, and to observe the structural differences between the modes.

As before, asymmetric simulations were started from an axisymmetric solution with a small asymmetric random perturbation added (typically of order \(10^{-4}\)). The flow was evolved to saturation, and the velocity fields saved for post-processing. Due to the computational expense of asymmetric wake flow computations, the azimuthal span of the bluff ring models was limited to the azimuthal wavelength of the desired shedding mode, as predicted by the stability analysis of Sheard et al. (2003).

Simulations were performed to capture each of the Modes A, B and C predicted by stability analysis (Sheard et al. 2003). Bluff rings of aspect ratio \(Ar = 5, 10, 20\) and 40 were simulated. The aspect ratios and the corresponding critical Reynolds numbers and azimuthal mode numbers for each asymmetric mode are listed in table 1.

Three sets of simulations have been performed to capture Modes A, B and C. Asymmetric flow visualisations of the resulting wake structures are analysed in §4.5.1, §4.5.2 and §4.5.3 for Modes A, B and C, respectively.

4.5.1. Asymmetric Structure of the Mode A Wake

Some examples of the asymmetric wakes simulated at Reynolds numbers above the critical Reynolds number predicted for the Mode A transition are provided in figure 14. The azimuthal span of each simulation was approximately \(4d\), with an azimuthal mode number corresponding to the predicted mode number of the Mode A instability for these aspect ratios from stability analysis (Sheard et al. 2003).
G. J. Sheard, M. C. Thompson and K. Hourigan

\begin{figure}[h]
\centering
(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d)
\includegraphics[width=\textwidth]{figure14}
\caption{Vortex structure of the asymmetric wakes of bluff rings computed over an azimuthal span of approximately 4d. The $\text{Ar} = 5$ ring is shown in (a), the $\text{Ar} = 10$ ring is shown in (b), (c) shows the $\text{Ar} = 20$ ring, and the $\text{Ar} = 40$ ring is illustrated in (d). Dark and light contours represent positive and negative streamwise vorticity respectively. A pressure of \textasciitilde-0.1 is represented by a translucent isosurface to indicate location of the vortex street. The bluff ring span is located at the upper right corner of each frame, and flow is towards the lower left corner. Note how part (a) shows two azimuthal periods of the faster-growing Mode C wake instead of the Mode A wake structures.}
\end{figure}

The plots in figures 14(b–d) all appear similar to the Mode A wake structures observed behind the circular cylinder both from experiment (Williamson 1988b, 1996) and numerically (Thompson et al. 1994, 1996; Henderson 1997). In each case, the wake consists of streamwise vortical structures stretched between the vortex cores. The spatio-temporal symmetry of these wakes is consistent with linear stability analysis predictions of the Mode A wake for the circular cylinder (Barkley & Henderson 1996), and for bluff rings (Sheard et al. 2003). A $1T$ symmetry is observed (the instantaneous wake structure is identical from one period $(T)$ to the next. Furthermore the location of the asymmetric structures obeys a qualitative spatio-temporal symmetry by which the distribution of asymmetric structures above and below the ring cross-section corresponds to the distribution of opposite sign asymmetric structures below and above the wake half a period later. The curvature of the bluff ring span breaks the quantitative nature of this symmetry from circular cylinder studies (Barkley & Henderson 1996), as the location and strength of the vortical structures inside and outside the ring cross-section are not quite equal.

The wake of the $\text{Ar} = 5$ ring in figure 14(a) differs from the other wakes in both symmetry and azimuthal wavelength. A $2T$ symmetry is observed (the instantaneous
asymmetric wake is self-similar over two periods of vortex shedding). Furthermore the azimuthal wavelength of the wake structures is half of the simulated domain (approximately 2d). These observations are consistent with the predictions of the Mode C wake from linear stability analysis of bluff rings (Sheard et al. 2003). Furthermore it is in agreement with the intermediate-wavelength Mode observed in the wake of perturbed circular cylinders (Zhang et al. 1995) and the unperturbed wake of square-section cylinders (Robichaux et al. 1999). The observation of this Mode C wake is consistent as table 1 shows the critical Reynolds number for the Mode C instability is lower than the critical Reynolds number for the Mode A instability for the \( Ar = 5 \) ring. The larger aspect ratio rings show that the Mode A instability dominates the asymmetric wake structure in agreement with the stability analysis prediction (Sheard et al. 2003).

4.5.2. Asymmetric Structure of the Mode B Wake

The azimuthal wavelength of the Mode B instability found for the circular cylinder is approximately 0.82d (Barkley & Henderson 1996). This wavelength corresponds to the wavelengths predicted by stability analysis on bluff rings (Sheard et al. 2003). By restricting the azimuthal span of the asymmetric simulations, only the Mode B instability will be permitted to grow, as it has the shortest azimuthal wavelength of the three asymmetric vortex shedding modes.

The isosurfaces plots in figure 15 show the asymmetric wake structures obtained at Reynolds numbers above the predicted critical Reynolds number for the Mode B instability for various bluff ring aspect ratios. The structures are similar in each case, and all satisfy the 1T spatio-temporal symmetry predicted for the corresponding Mode B instability of the circular cylinder wake (Barkley & Henderson 1996).

The asymmetric structures of the bluff ring wakes in figure 15 are dominated by streamwise vortices in the braid region between the vortex cores. Similar structures have been
observed in experimental dye visualisation of the Mode B instability for the circular cylinder (Williamson 1988b, 1996). Numerical simulations have also elucidated similar structures for the circular cylinder (Thompson et al. 1994, 1996; Henderson 1997). Further numerical observations on the wake of the square cylinder (Robichaux et al. 1999) and the perturbed circular cylinder wake (Zhang et al. 1995) indicate that the wakes observed here in figure 15 are analogous to the Mode B wakes found for various cylindrical geometries in terms of spatio-temporal symmetry, azimuthal wavelength, and wake structure.

4.5.3. Asymmetric Structure of the Mode C Wake

The Mode C instability was predicted to have an azimuthal wavelength of between 1.6d and 1.7d from stability analysis of bluff ring wakes (Sheard et al. 2003). Instabilities of a similar azimuthal span have been shown to occur in the wakes of both square-section cylinders (Robichaux et al. 1999), and circular cylinder wakes perturbed by a trip-wire (Zhang et al. 1995). To date no evidence of this mode has been shown for the unperturbed circular cylinder, neither in experiment (Williamson 1996), nor numerical stability analysis (Barkley & Henderson 1996).

Williamson (1988b) showed that the two discontinuities in the Strouhal number profile for the circular cylinder wake with increasing Reynolds number were due to the respective emergence of the Mode A and Mode B instabilities in the vortex street. Numerical stability analysis on the circular cylinder wake (Barkley & Henderson 1996) showed that no real Floquet modes for azimuthal wavelengths in the range 1.2d < \lambda < 2.5d could be obtained. Only complex conjugate pairs of Floquet multipliers were obtained over this range, and the real component of these multipliers remained far below the neutral stability threshold \mu = 1.

The Mode C type instabilities predicted by Zhang et al. (1995) and Robichaux et al. (1999) showed a subharmonic 2T spatio-temporal symmetry, where the vortex pattern of the instability was repeated each period with an opposite sign to the previous period. The Mode C wakes are computed here for the same aspect ratios as the simulations in §4.5.1 and §4.5.2 for capturing the Mode A and Mode B wake structures, respectively.

Isosurface plots of the resulting wake structures can be seen in figure 16. An interesting observation can be made from the isosurfaces plots in figure 16. The simulated azimuthal span of approximately 1.6d to 1.7d is twice the azimuthal span of the Mode B instability. Furthermore, the Mode B instability has a lower critical Reynolds number than the Mode C instability for the bluff rings with aspect ratios Ar = 20 and Ar = 40 (figures 16(c) and 16(d) respectively). The plots of these aspect ratios clearly show the presence of Mode B type wake structures in the near wake behind the ring cross-section. As expected, the azimuthal span of the Mode B structures is half that of the Mode C structures that comprise the far wakes of these aspect ratios, and the entire wakes of the rings with aspect ratios Ar = 5 (figure 16(a)) and Ar = 10 (figure 16(b)).

The time histories measured from a point in the wake of the bluff rings showed that the saturated wake following the Mode C instability did indeed adopt the subharmonic behaviour predicted from stability analysis of the bluff ring wakes (Sheard et al. 2003) and the numerical work on the Mode C instability for other geometries (Robichaux et al. 1999; Zhang et al. 1995). Despite periodicity being maintained over two shedding cycles, the period of odd cycles lengthened by about 5%, and the period of even cycles shortened by a corresponding amount. This effect is caused by the asymmetry of the vorticity distribution in the wake between even and odd shedding cycles displacing the underlying vortex street.
4.6. Landau modelling of the Asymmetric Vortex Shedding Modes

The non-linear transition behaviour of the asymmetric vortex shedding Modes A, B and C will be investigated in the following sections §4.6.1, §4.6.2 and §4.6.3 respectively by application of the Landau model as per §4.4.

The Landau model has been applied to the non-linear transition to Mode A and Mode B wakes for the straight circular cylinder (Henderson 1997). He showed the Mode A transition to occur through a subcritical bifurcation, and the Mode B transition to occur through a supercritical bifurcation. We verify that the criticality of circular cylinder modes agree with the criticality of the Mode A and Mode B transitions for bluff ring wakes, and attempt to ascertain the criticality of the Mode C transition.

4.6.1. Criticality of the Mode A Transition

The Mode A transition occurs through a subcritical bifurcation in the wake of a circular cylinder, and the same criticality is found for the Mode A transition of bluff ring wakes. The Mode A transition was computed for a ring with $Ar = 20$. The growth of the envelope of oscillation of the instability is plotted in figure 17. The positive gradient near the $y$ axis, and the non-linearity of the plot in figure 17(b) validate the prediction of a subcritical transition for the Mode A instability. The growth rate of $\sigma = 0.024$ is determined, and a non-linear profile is observed.

4.6.2. Criticality of the Mode B Transition

Simulations of the $Ar = 20$ ring at a Reynolds number $Re = 270$ were performed to determine the criticality of the Mode B wake. The behaviour of the Mode B instability is shown in the plots in figure 18. The transition to a Mode B wake is through a supercritical bifurcation, according to the plots in figure 18. The growth rate of $\sigma = 0.0525$
Figure 17. (a) Left: Growth and saturation of the envelope of the asymmetric transient corresponding to the Mode A instability in the wake of a bluff ring with aspect ratio $Ar = 20$ at $Re = 205$. (b) Right: A plot of the amplitude derivative versus the square of the amplitude, showing subcritical behaviour.

Figure 18. (a) Left: Growth and saturation of the envelope of the asymmetric transient corresponding to the Mode B instability of the wake of a bluff ring with aspect ratio $Ar = 20$ at $Re = 270$. (b) Right: A plot of the amplitude derivative versus the square of the amplitude showing supercritical behaviour. The dotted line is an extrapolation of the linear profile.

is determined, and a largely linear profile with negative gradient observed. This finding agrees with the determination of the criticality of the Mode B instability for the circular cylinder (Henderson 1997).

4.6.3. Criticality of the Mode C Transition

The non-linear behaviour of the Mode C transition has been observed for the $Ar = 5$ ring at $Re = 170$. The charts in figure 19 show the non-linear saturation of the Mode C instability for the $Ar = 5$ ring, and the negative gradient in figure 19(b) clearly indicates that the Mode C transition occurs through a supercritical bifurcation. A growth rate of $\sigma = 0.0144$ is found, and a negative $l$-term is observed.

5. Conclusions

Asymmetric isosurface plots have been presented showing the wake structures that occur following both the regular and Hopf transitions for tori with aspect ratios $Ar \leq 3.9$. 
The predictions from a previous stability analysis study have been verified pertaining to the presence of three distinct transition modes in the aspect ratio regime $0 \leq Ar \leq 3.9$, and the transition Reynolds numbers. These transitions are: the regular transition, Mode I, occurring in the aspect ratio range $0 \leq Ar < 1.6$; the Hopf transition, Mode II, occurring over the range $1.6 \leq Ar \leq 1.7$; and the regular transition, Mode III, occurring over the range $1.7 < Ar < 4$.

The Mode I transition is shown to be analogous to the regular transition of the sphere wake to an asymmetric state, and is characterised by the classic “double-threaded wake” observed for a sphere. The azimuthal symmetry of the Mode I transition is $m = 1$. The Mode II transition is shown to consist of a very low frequency oscillating asymmetric wake, again with azimuthal symmetry $m = 1$. The Mode III wake is dominated by asymmetric vortical structures in the near-wake region in the vicinity of the recirculation bubble behind the ring cross-section. The azimuthal symmetry of the Mode III transition is $m = 1$ for $Ar = 2$, and $m = 2$ for $Ar = 3$. The secondary transitions which occur after the Mode I and Mode III regular transitions were found to be Hopf bifurcations, with the resulting wakes consisting of hairpin style vortex shedding analogous to the unsteady wake behind the sphere. The azimuthal symmetry of these wakes was $m = 1$ for all aspect ratios in the range $0 \leq Ar \leq 2$, and $m = 2$ for $Ar = 3$.

The non-linear behaviour of the wake following transition has been modelled by the Landau equation. This enabled the criticality of the transition modes to be determined. The Mode I transition was found to occur through a supercritical bifurcation of the steady axisymmetric wake (as per the sphere wake). The Mode III transition is a subcritical transition, indicating hysteretic behaviour in the vicinity of the transition.

The Hopf transitions following both the Mode I and Mode III transitions were supercritical, and the Mode II transition was also found to occur through a supercritical bifurcation.

The asymmetric wakes that occur as a result of the Mode A, Mode B and Mode C transitions were presented as isosurfaces plots of streamwise vorticity and pressure. The Mode A and Mode B wakes were found to be analogous to the corresponding wakes for the circular cylinder. The existence of the Mode C transition predicted for the tori by stability analysis was verified by asymmetric simulation. For each mode the observed spatio-temporal symmetry was the same as that for the corresponding Mode C in the wakes.
of circular and square cylinders. Finally, Landau modelling enabled the criticality of the asymmetric vortex shedding modes to be determined. The analysis verified the previous findings for the circular cylinder Mode A and Mode B transitions, being subcritical and supercritical respectively. The Mode C transition was found to occur through a supercritical bifurcation.

In summary, the non-linear behaviour of the Hopf transition in the wake of bluff rings shows the bifurcation to occur through a supercritical transition for all aspect ratios. Furthermore, the primary asymmetric instability of the wake of bluff rings occurs through a supercritical bifurcation for all aspect ratios except those pertaining to the Mode III transition ($1.7 < Ar \leq 3.9$), and the Mode A transition ($Ar \geq 8$), where subcritical asymmetric transitions occur. Hence, the wake behind toroidal geometries will exhibit no hysteretic behaviour through the primary Hopf transition. In contrast, the primary asymmetric transition will exhibit hysteresis for the majority of toroidal aspect ratio configurations.

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